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Jurek Czyzowicz[1,⋆], David Ilcinkas[2,⋆⋆], Arnaud Labourel[2,⋆⋆,⋆⋆⋆], and Andrzej Pelc[1,†]

[1] Département d'informatique, Université du Québec en Outaouais, Gatineau, Québec J8X 3X7, Canada. E-mails: jurek@uqo.ca, pelc@uqo.ca

[2] LaBRI, CNRS & Université de Bordeaux, 33405 Talence, France. E-mails: david.ilcinkas@labri.fr, labourel.arnaud@gmail.com,

Abstract. A mobile robot represented by a point moving in the plane has to explore an unknown flat terrain with impassable obstacles. Both the terrain and the obstacles are modeled as arbitrary polygons. We consider two scenarios: the *unlimited vision*, when the robot situated at a point p of the terrain explores (sees) all points q of the terrain for which the segment pq belongs to the terrain, and the *limited vision*, when we require additionally that the distance between p and q be at most 1. All points of the terrain (except obstacles) have to be explored and the performance of an exploration algorithm, called its complexity, is measured by the length of the trajectory of the robot.

For unlimited vision we show an exploration algorithm with complexity $O(P + D\sqrt{k})$, where P is the total perimeter of the terrain (including perimeters of obstacles), D is the diameter of the convex hull of the terrain, and k is the number of obstacles. We do not assume knowledge of these parameters. We also prove a matching lower bound showing that the above complexity is optimal, even if the terrain is known to the robot. For limited vision we show exploration algorithms with complexity $O(P+A+\sqrt{Ak})$, where A is the area of the terrain (excluding obstacles). Our algorithms work either for arbitrary terrains, if one of the parameters A or k is known, or for c-fat terrains, where c is any constant (unknown to the robot) and no additional knowledge is assumed. (A terrain $\mathcal{T}$ with obstacles is c-fat if $R/r \leq c$, where R is the radius of the smallest disc containing $\mathcal{T}$ and r is the radius of the largest disc contained in $\mathcal{T}$.) We also prove a matching lower bound $\Omega(P+A+\sqrt{Ak})$ on the complexity of exploration for limited vision, even if the terrain is known to the robot.

keywords: mobile robot, exploration, polygon, obstacle.

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1 Introduction

The background and the problem. Exploring unknown terrains by mobile robots has important applications when the environment is dangerous or of difficult access for humans. Such is the situation when operating in nuclear plants or cleaning toxic wastes, as well as in the case of underwater or extra-terrestrial operations. In many cases a robot must inspect an unknown terrain and come back to its starting point. Due to energy and cost saving requirements, the length of the robot's trajectory should be minimized.

We model the exploration problem as follows. The terrain is represented by an arbitrary polygon $\mathcal{P}_0$ with pairwise disjoint polygonal obstacles $\mathcal{P}_1, ..., \mathcal{P}_k$, included in $\mathcal{P}_0$, i.e., the terrain is $\mathcal{T} = \mathcal{P}_0 \setminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k)$. We assume that borders of all polygons $\mathcal{P}_i$ belong to the terrain. The robot is modeled as a point moving along a polygonal line inside the terrain. It should be noted that the restriction to polygons is only to simplify the description, and all our results hold in the more general case where polygons are replaced by bounded subsets of the plane homeotopic with a disc (i.e., connected and without holes) and regular enough to have well-defined area and boundary length. Every point of the trajectory of the robot is called *visited*. We consider two scenarios: the *unlimited vision*, when the robot visiting a point p of the terrain $\mathcal{T}$ *explores* (sees) all points q for which the segment pq is entirely contained in $\mathcal{T}$, and the *limited vision*, when we require additionally that the distance between p and q be at most 1. In both cases the task is to explore all points of the terrain $\mathcal{T}$. The cost of an exploration algorithm is the length of the trajectory of the robot, which should be as small as possible. The *complexity* of an algorithm is the order of magnitude of its cost. We assume that the robot does not know the terrain before starting the exploration, but it has unbounded memory and can record the portion of the terrain seen so far and the already visited portion of its trajectory.

Our results. For unlimited vision we show an exploration algorithm with complexity $O(P + D\sqrt{k})$, where P is the total perimeter of the terrain (including perimeters of obstacles), D is the diameter of the convex hull of the terrain, and k is the number of obstacles. We do not assume knowledge of these parameters. We also prove a matching lower bound for exploration of some terrains (even if the terrain is known to the robot), showing that the above complexity is worst-case optimal.

For limited vision we show exploration algorithms with complexity $O(P + A + \sqrt{Ak})$, where A is the area of the terrain[1]. Our algorithms work either for arbitrary terrains, if one of the parameters A or k is known, or for c-fat terrains, where c is any constant larger than 1 (unknown to the robot) and no additional knowledge is assumed. (A terrain $\mathcal{T}$ is c-fat if $R/r \leq c$, where R is the radius of

[1] Since parameters D, P, A are positive reals that may be arbitrarily small, it is important to stress that complexity $O(P + A + \sqrt{Ak})$ means that the trajectory of the robot is at most $c(P + A + \sqrt{Ak})$, for some constant c and *sufficiently large* values of P and A. Similarly for $O(P + D\sqrt{k})$. This permits to include, e.g., additive constants in the complexity, in spite of arbitrarily small parameter values.

the smallest disc containing $\mathcal{T}$ and r is the radius of the largest disc contained in $\mathcal{T}$.) We also prove a matching lower bound $\Omega(P + A + \sqrt{Ak})$ on the complexity of exploration, even if the terrain is known to the robot.

The main open problem resulting from our research is whether exploration with asymptotically optimal cost $O(P+A+\sqrt{Ak})$ can be performed in arbitrary terrains without *any* a priori knowledge. Another interesting open problem is whether such worst-case performance can be obtained by an $O(k)$-competitive algorithm. (Our algorithms are a priori not competitive.)

Related work. Exploration of unknown environments by mobile robots was extensively studied both for the unlimited and for the limited vision. Most of the research in this domain concerns the competitive framework, where the trajectory of the robot not knowing the environment is compared to that of the optimal exploration algorithm having full knowledge.

One of the most important works for unlimited vision is [8]. The authors gave a 2-competitive algorithm for rectilinear polygon exploration without obstacles. The case of non-rectilinear polygons (without obstacles) was also studied in [7, 15, 12] and competitive algorithms were given.

For polygonal environments with an arbitrary number of polygonal obstacles, it was shown in [8] that no competitive strategy exists, even if all obstacles are parallelograms. Later, this result was improved in [1] by giving a lower bound in $\Omega(\sqrt{k})$ for the competitive ratio of any on-line algorithm exploring a polygon with k obstacles. This bound remains true even for rectangular obstacles. On the other hand, there exists an algorithm with competitive ratio in $O(k)$ [7, 15]. Moreover, for particular shapes of obstacles (convex and with bounded aspect ratio) the optimal competitive ratio $\Theta(\sqrt{k})$ has been proven in [15].

Exploration of polygons by a robot with limited vision has been studied, e.g., in [9–11, 13, 14, 16]. In [9] the authors described an on-line algorithm with competitive ratio $1+3(\Pi S/A)$, where Π is a quantity depending on the perimeter of the polygon, S is the area seen by the robot, and A is the area of the polygon. The exploration in [9, 10] fails on a certain type of polygons, such as those with narrow corridors. In [11], the authors consider exploration in discrete steps. The robot can only explore the environment when it is motionless, and the cost of the exploration algorithm is measured by the number of stops during the exploration. In [13, 14], the complexity of exploration is measured by the trajectory length, but only terrains composed of identical squares are considered. In [16] the author studied off-line exploration of the boundary of a terrain with limited vision.

An experimental approach was used in [2] to show the performance of a greedy heuristic for exploration in which the robot always moves to the frontier between explored and unexplored area. Practical exploration of the environment by an actual robot was studied, e.g., in [6, 19]. In [19], a technique is described to deal with obstacles that are not in the plane of the sensor. In [6] landmarks are used during exploration to construct the skeleton of the environment.

Navigation is a closely related task which consists in finding a path between two given points in a terrain with unknown obstacles. Navigation in a $n \times n$ square containing rectangular obstacles aligned with sides of the square was

considered in [3–5, 18]. It was shown in [3] that the navigation from a corner to the center of a room can be performed with a competitive ratio $O(\log n)$, only using tactile information (i.e., the robot modeled as a point sees an obstacle only when it touches it). No deterministic algorithm can achieve better competitive ratio, even with unlimited vision [3]. For navigation between any pair of points, there is a deterministic algorithm achieving a competitive ratio of $O(\sqrt{n})$ [5]. No deterministic algorithm can achieve a better competitive ratio [18]. However, there is a randomized approach performing navigation with a competitive ratio of $O(n^{\frac{4}{9}} \log n)$ [4]. Navigation with little information was considered in [20]. In this model, the robot cannot perform localization nor measure any distances or angles. Nevertheless, the robot is able to learn the critical information contained in the classical shortest-path roadmap and perform locally optimal navigation.

2 Unlimited vision

Let S be a smallest square in which the terrain $\mathcal{T}$ is included. Our algorithm constructs a *quadtree decomposition* of S. A quadtree is a rooted tree with each non-terminal node having four children. Each node of the quadtree corresponds to a square. The children of any non-terminal node v correspond to four identical squares obtained by partitioning the square of v using its horizontal and vertical symmetry axes. This implies that the squares of the terminal nodes form a partition of the root[2]. More precisely,

1. $\{S\}$ is a quadtree decomposition of S
2. If $\{S_1, S_2, \ldots, S_j\}$ is a quadtree decomposition of S, then $\{S_1, S_2, \ldots, S_{i-1}, S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}, S_{i+1}, \ldots, S_j\}$, where $S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}$ form a partition of S_i using its vertical and horizontal symmetry axes, is a quadtree decomposition of S

The trajectory of the robot exploring $\mathcal{T}$ will be composed of parts which will follow the boundaries of $\mathcal{P}_i$, for $0 \leq i \leq k$, and of straight-line segments, called *approaching segments*, joining the boundaries of $\mathcal{P}_i$, $0 \leq i \leq k$. Obviously, the end points of an approaching segment must be visible from each other. The quadtree decomposition will be dynamically constructed in a top-down manner during the exploration of $\mathcal{T}$. At each moment of the exploration we consider the set $\mathcal{Q}_\mathcal{S}$ of all squares of the current quadtree and the set $\mathcal{Q}_\mathcal{T}$ of squares being the terminal nodes of the current quadtree. We will also construct dynamically a bijection $f : \{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_k\} \longrightarrow \mathcal{Q}_\mathcal{S} \setminus \mathcal{Q}_\mathcal{T}$.

When a robot moves along the boundary of some polygon $\mathcal{P}_i$, it may be in one of two possible modes: the *recognition mode* - when it goes around the entire boundary of a polygon without any deviation, or in the *exploration mode* - when, while moving around the boundary, it tries to detect (and approach) new obstacles. When the decision to approach a new obstacle is made at some point

[2] In order to have an exact partition we assume that each square of the quadtree partition contains its East and South edges but not its West and North edges.

r of the boundary of $\mathcal{P}_i$ the robot moves along an approaching segment to reach the obstacle, processes it by a recursive call, and (usually much later), returning from the recursive call, it moves again along this segment in the opposite direction in order to return to point r and to continue the exploration of $\mathcal{P}_i$. However, some newly detected obstacles may not be immediately approached. We say that, when the robot is in position r, an obstacle $\mathcal{P}_j$ is *approachable*, if there exists a point $q \in \mathcal{P}_j$, belonging to a square $S_t \in \mathcal{Q}_\mathcal{T}$ of diameter $D(S_t)$ such that $|rq| \leq 2D(S_t)$. It is important to state that if exactly one obstacle becomes approachable at moment t, then it is approached immediately and if more than one obstacle become approachable at a moment t, then one of them (chosen arbitrarily) is approached immediately and the others are approached later, possibly from different points of the trajectory. Each time a new obstacle is visited by the robot (i.e., all the points of its boundary are visited in the recognition mode) the terminal square of the current quadtree containing the first visited point of the new obstacle is partitioned. This square is then associated to this obstacle by function f. The trajectory of the robot is composed of three types of sections: *recognition sections*, *exploration sections* and *approaching sections*. The boundary of each polygon will be traversed twice: first time contiguously during a recognition section and second time through exploration sections, which may be interrupted several times in order to approach and visit newly detected obstacles. We say that an obstacle is *completely explored*, if each point on the boundary of this obstacle has been traversed by an exploration section. We will prove that the sum of the lengths of the approaching sections is $O(D\sqrt{k})$.

Algorithm `ExpTrav` (polygon R, starting point r^* on the boundary of R)
1 Make a recognition traversal of the boundary of R
2 Partition square $S_t \in \mathcal{Q}_\mathcal{T}$ containing r^* into four identical squares
3 $f(R) := S_t$
4 **repeat**
5 Traverse the boundary of R until, for the current position r, there exists a visible point q of a new obstacle Q belonging to square $S_t \in \mathcal{Q}_\mathcal{T}$, such that $|rq| \leq 2D(S_t)$
6 Traverse the segment rq
7 ExpTrav(Q, q)
8 Traverse the segment qr
9 **until** R is completely explored

Before the initial call of `ExpTrav`, the robot reaches a position r_0 at the boundary of the polygon $\mathcal{P}_0$. This is done as follows. At its initial position v, the robot chooses an arbitrary half-line α which it follows as far as possible. When it hits the boundary of a polygon $\mathcal{P}$, it traverses the entire boundary of $\mathcal{P}$. Then, it computes the point u which is the farthest point from v in $\mathcal{P} \cap \alpha$. It goes around $\mathcal{P}$ until reaching u again and progresses on α, if possible. If this is impossible, the robot recognizes that it went around the boundary of $\mathcal{P}_0$ and it is positioned on this boundary. It initialises the quadtree decomposition to a smallest square S containing $\mathcal{P}_0$. This square is of size $O(D(\mathcal{P}_0))$. The length of the above walk is less than $3P$.

Lemma 1. *Algorithm* `ExpTrav` *visits all boundary points of all obstacles of the terrain* $\mathcal{T}$*.*

Lemma 2. *Function* f *is a bijection from* $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_k\}$ *to* $\mathcal{Q}_\mathcal{S} \setminus \mathcal{Q}_\mathcal{T}$*, where* $\mathcal{Q}_\mathcal{S}$ *and* $\mathcal{Q}_\mathcal{T}$ *correspond to the final quadtree decomposition produced by Algorithm* `ExpTrav`*.*

Lemma 3. *For any quadtree* T*, rooted at a square of diameter* D *and having* x *non-terminal nodes, the sum* $\sigma(T)$ *of diameters of these nodes is at most* $2D\sqrt{x}$*.*

Theorem 1. *Algorithm* `ExpTrav` *explores the terrain* $\mathcal{T}$ *of perimeter* P *and convex hull diameter* D *with* k *obstacles in time* $O(P + D\sqrt{k})$*.*

Proof. Take an arbitrary point p inside $\mathcal{T}$ and a ray outgoing from p in an arbitrary direction. This ray reaches the boundary of $\mathcal{T}$ at some point q. Since, by Lemma 1 point q was visited by the robot, p was visible from q during the robot's traversal, and hence p was explored.

To prove the complexity of the algorithm, observe that the robot traverses twice the boundary of each polygon of $\mathcal{T}$, once during its recognition in step 1 and the second time during the iterations of step 5. Hence the sum of lengths of the recognition and exploration sections is $2P$. The only other portions of the trajectory are produced in steps 6 and 8, when the obstacles are approached and returned from. According to the condition from step 5, an approaching segment is traversed in step 6 only if its length is shorter than twice the diameter of the associated square. If $k = 0$ then the sum of lengths of all approaching segments is 0, due to the fact that exploration starts at the external boundary of the terrain. In this case the length of the trajectory is at most $5P$ (since the length of at most $3P$ was traversed before the initial call). Hence we may assume that $k > 0$. By Lemma 2 each obstacle is associated with a different non-terminal node of the quadtree and the number x of non-terminal nodes of the quadtree equals $k + 1$. Hence the sum of lengths of all approaching segments is at most $2\sigma(T)$. By Lemma 3 we have $\sigma(T) \leq 2D\sqrt{x} = 2D\sqrt{k+1}$, hence the sum of lengths of approaching segments is at most $2\sigma(T) \leq 4D\sqrt{k+1} \leq 4D\sqrt{2k} \leq 6D\sqrt{k}$. Each segment is traversed twice, so the total length of this part of the trajectory is at most $12D\sqrt{k}$. It follows that the total length of the trajectory is at most $5P + 12D\sqrt{k}$. □

Theorem 2. *Any algorithm for a robot with unlimited visibility, exploring polygonal terrains with* k *obstacles, having total perimeter* P *and the convex hull diameter* D*, produces trajectories in* $\Omega(P + D\sqrt{k})$ *in some terrains, even if the terrain is known to the robot.*

Proof. In order to prove the lower bound, we show two families of terrains: one for which $P \in \Theta(D)$ (P cannot be smaller), D and k are unbounded and still the exploration cost is $\Omega(D\sqrt{k})$, and the other in which P is unbounded, D is arbitrarily small, $k = 0$ and still the exploration cost is $\Omega(P)$. Consider the terrain from Figure 1(a) where k identical tiny obstacles are distributed evenly at the $\sqrt{k} \times \sqrt{k}$ grid positions inside a square of diameter D. The distance between

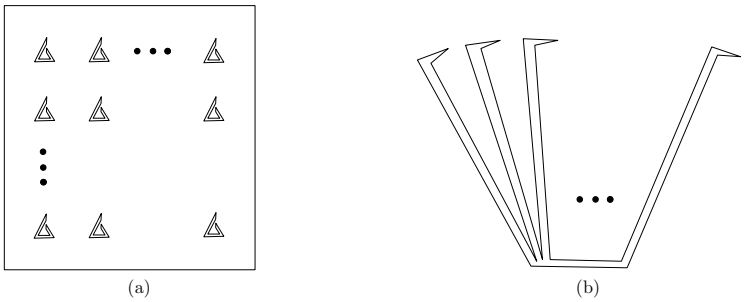

Fig. 1. Lower bound for unlimited visiblity

obstacles is at least $\frac{D\sqrt{2}}{2(\sqrt{k}+1)} - \epsilon$ where $\epsilon > 0$ may be as small as necessary by choosing obstacles sufficiently small. The obstacles are such that to explore the small area inside the convex hull of the obstacle the robot must enter this convex hull. Since each such area must be explored, the trajectory of the robot must be of size at least $(k-1)\left(\frac{D\sqrt{2}}{2(\sqrt{k}+1)} - \epsilon\right)$, which is clearly in $\Omega(D\sqrt{k})$. Note that the perimeter P is in $\Theta(D)$.

The terrain from Fig. 1(b) is a polygon of arbitrarily small diameter (without obstacles), whose exploration requires a trajectory of size $\Omega(P)$, where P is unbounded (as the number of "corridors" can be unbounded). Indeed, each "corridor" must be traversed almost completely to explore points at its end. The two families of polygons from Fig. 1 lead to the $\Omega(P + D\sqrt{k})$ lower bound. □

3 Limited vision

In this section we assume that the vision of the robot has range 1. The following algorithm is at the root of all our positive results on exploration with limited vision. The idea of the algorithm is to partition the environment into small parts called *cells* (of diameter at most 1) and to visit them using a depth-first traversal. The local exploration of cells can be performed using Algorithm `ExpTrav`, since the vision inside each cell is not limited by the range 1 of the vision of the robot. The main novelty of our exploration algorithm is that the robot completely explores *any* terrain. This should be contrasted with previous algorithms with limited visibility, e.g. [9, 10, 13, 14] in which only a particular class of terrains with obstacles is explored, e.g., terrains without narrow corridors or terrains composed of complete identical squares. This can be done at cost $O(A)$. Our lower bound shows that exploration complexity of arbitrary terrains depends on the perimeter and the number of obstacles as well. The complete exploration of arbitrary terrains achieved by our algorithm significantly complicates both the exploration process and its analysis.

Our algorithms $\texttt{LET}_\texttt{A}$ and $\texttt{LET}_\texttt{k}$, and the tourist algorithm described in [15] share a similar approach to exploration, i.e., using several square decompositions

of the terrain with different side lengths to figure out the characteristics of the terrain and achieve efficient exploration. However, our algorithms differ from the tourist algorithm in two important ways : (1) the exploration of the inside of each square is done with an optimal algorithm (See section 2) instead of a greedy one and (2) the limited visibility forces an upper bound on the side of the square (significantly complicating $\texttt{LET}_\texttt{k}$). More importantly, due to the numerous differences between our model and the one of [15], the analyses of the complexities of the algorithms are unrelated.

Algorithm `LimExpTrav` (*LET*, for short)
INPUT: A point s inside the terrain $\mathcal{T}$ and a positive real $F \leq \sqrt{2}/2$.
OUTPUT: An exploration trajectory of $\mathcal{T}$, starting and ending at s.
Tile the area with squares of side F, such that s is on the boundary of a square. The connected regions obtained as intersections of $\mathcal{T}$ with each tile are called *cells*. For each tile S, maintain a quadtree decomposition Q_S initially set to $\{S\}$. Then, arbitrarily choose one of the cells containing s to be the starting cell C and call `ExpCell`(C, s).

Procedure `ExpCell`(current cell C, starting point $r^* \in C$)
1 Record C as visited
2 `ExpTrav`(C,r^*) using the quadtree decomposition Q_S; S is the tile s.t. $C \subseteq S$
3 **repeat**
4 Traverse the boundary of C until the current position r belongs to an unvisited cell U
5 `ExpCell`(U, r)
(if r is in several unvisited cells, choose arbitrarily one to be processed)
6 **until** the boundary of C is completely traversed

It is worth to note that, at the beginning of the exploration of the first cell belonging to a tile S, the quadtree of this tile is set to a single node. However, at the beginning of explorations of subsequent cells belonging to S, the quadtree of S may be different. So the top-down construction of this quadtree may be spread over the exploration of many cells which will be visited at different points in time.

Consider a tile T and a cell $C \subseteq T$. Let A_C be the area of C and B_C the length of its boundary. Let P_C be the length of the part of the boundary of C included in the boundary of the terrain $\mathcal{T}$, and let R_C be the length of the remaining part of the boundary of C, i.e., $R_C = B_C - P_C$.

Lemma 4. *There is a positive constant c, such that $R_C \leq c(A_C/F + P_C)$, for any cell C.*

The following is the key lemma for all upper bounds proved in this section. Let $\mathcal{S} = \{T_1, T_2, \ldots, T_n\}$ be the set of tiles with non-empty intersection with $\mathcal{T}$ and $\mathcal{C} = \{C_1, C_2, \cdots, C_m\}$ be the set of cells that are intersections of tiles from $\mathcal{S}$ with $\mathcal{T}$. For each $T \in \mathcal{S}$, let k_T be the number of obstacles of $\mathcal{T}$ entirely contained in T.

Lemma 5. *For any $F \leq \sqrt{2}/2$, Algorithm LET explores the terrain $\mathcal{T}$ of area A and perimeter P, using a trajectory of length $O(P + A/F + F\sum_{i=1}^{n}\sqrt{k_{T_i}})$.*

Proof. First, we show that Algorithm *LET* explores the terrain $\mathcal{T}$. Consider the graph G whose vertex set is $\mathcal{C}$ and edges are the pairs $\{C, C'\}$ such that C and C' have a common point at their boundaries. The graph G is connected, since $\mathcal{T}$ is connected. Note that for any cell C and point r on the boundary of C, `ExpTrav` on C and r and thus `ExpCell` on C and r starts and ends on r. Therefore, Algorithm *LET* performs a depth first traversal of graph G, since during the execution of `ExpCell`$(C, \dots)$, procedure `ExpCell`$(U, \cdots)$ is called for each unvisited cell U adjacent to C. Hence, `ExpCell`$(C, \dots)$ is called for each cell $C \in \mathcal{C}$, since G is connected. During the execution of `ExpCell`(C, r), C is completely explored by `ExpTrav`(C,r) by the same argument as in the proof of Lemma 1, since the convex hull diameter of C is less than one.

It remains to show that the length of the *LET* trajectory is $O(P + A/F + F\sum_{i=1}^{n}\sqrt{k_{T_i}})$. For each $j = 1, \dots, m$, the part of the *LET* trajectory inside the cell C_j is produced by the execution of `ExpCell`$(C_j, \dots)$. In step 2 of `ExpCell`$(C_j, \dots)$, the robot executes `ExpTrav` with $D = \sqrt{2}F$ and $P = P_{C_j} + R_{C_j}$. The sum of lengths of recognition and exploration sections of the trajectory in C_j is at most $2(P_{C_j} + R_{C_j})$. The sum of lengths of approaching sections of the trajectory in T_i is at most $6\sqrt{2}F\sqrt{k_{T_i}}$ and each approaching section is traversed twice (cf. proof of Theorem 1). In step 3 of `ExpCell`$(C_j, \dots)$, the robot only makes the tour of the cell C_j, hence the distance traveled by the robot is at most $P_{C_j} + R_{C_j}$. It follows that:

$$
\begin{aligned}
|\, LET \,| &\le 3\sum_{j=1}^{m}(P_{C_j} + R_{C_j}) + 12\sqrt{2}F\sum_{i=1}^{n}\sqrt{k_{T_i}} \\
&\le 3\sum_{j=1}^{m}((1+c)P_{C_j} + cA_{C_j}/F) + 12\sqrt{2}F\sum_{i=1}^{n}\sqrt{k_{T_i}} \quad \text{by Lemma 4} \\
&\le 3(c+1)P + 3cA/F + 12\sqrt{2}F\sum_{i=1}^{n}\sqrt{k_{T_i}}.
\end{aligned}
$$

□

In view of Lemma 5, exploration of a particular class of terrains can be done at a cost which will be later proved optimal.

Theorem 3. *Let $c > 1$ be any constant. Exploration of a c-fat terrain of area A, perimeter P and with k obstacles can be performed using a trajectory of length $O(P + A + \sqrt{Ak})$ (without any a priori knowledge).*

Proof. The robot executes Algorithm *LET* with $F = \sqrt{2}/2$. By Lemma 5, the total cost is $O(P+A+\sum_{i=1}^{n}\sqrt{k_{T_i}})$. Recall that n is the number of tiles that have non-empty intersection with the terrain. We have $\sum_{i=1}^{n}\sqrt{k_{T_i}} \le \sum_{i=1}^{n}\sqrt{\frac{k}{n}} = \sqrt{nk}$. Hence, it remains to show that $n = O(A)$ to prove that the cost is $O(P + A + \sqrt{Ak})$. By definition of a c-fat terrain, there is a disk D_1 of radius r included in the terrain and a disk D_2 of radius R that contains the terrain, such that

$\frac{R}{r} \leq c$. There are $\Theta(r^2)$ tiles entirely included in D_1 and hence in the terrain. So, we have $A = \Omega(r^2)$. $\Theta(R^2)$ tiles are sufficient to cover D_2 and hence the terrain. So $n = O(R^2)$. Hence, we obtain $n = O(A)$ in view of $R \leq cr$. □

Consider any terrain $\mathcal{T}$ of area A, perimeter P and with k obstacles. We now turn attention to the exploration problem if some knowledge about the terrain is available a priori. Notice that if A and k are known before the exploration, Lemma 5 implies that Algorithm *LET* executed for $F = \min\{\sqrt{A/k}, \sqrt{2}/2\}$ explores *any* terrain at cost $O(A + P + \sqrt{Ak})$. (Indeed, if $F = \sqrt{A/k}$ then $A/F = \sqrt{Ak}$ and $kF = \sqrt{Ak}$, while $F = \sqrt{2}/2$ implies $A/F = \Theta(A)$ and $kF = O(A)$.) This cost will be later proved optimal. It turns out that a much more subtle use of Algorithm *LET* can guarantee the same complexity assuming only knowledge of A *or* k. We present two different algorithms depending on which value, A or k, is known to the robot. Both algorithms rely on the same idea. The robot executes Algorithm *LET* with some initial value of F until either the terrain is completely explored, or a certain stopping condition, depending on the algorithm, is satisfied. This execution constitutes the first stage of the two algorithms. If exploration was interrupted because of the stopping condition, then the robot proceeds to a new stage by executing Algorithm *LET* with a new value of F. Values of F decrease in the first algorithm and increase in the second one. The exploration terminates at the stage when the terrain becomes completely explored, while the stopping condition is never satisfied. In each stage the robot is oblivious of the previous stages, except for the computation of the new value of F that depends on the previous stage. This means that in each stage exploration is done "from scratch", without recording what was explored in previous stages. In order to test the stopping condition in a given stage, the robot maintains the following three values: the sum A^* of areas of explored cells, updated after the execution of `ExpTrav` in each cell; the length P^* of the boundary traversed by the robot, continuously updated when the robot moves along a boundary for the first time (i.e., in the recognition mode); and the number k^* of obstacles approached by the robot, updated when an obstacle is approached. The values of A^*, P^* and k^* at the end of the i-th stage are denoted by A_i, P_i and k_i, respectively. Let F_i be the value of F used by Algorithm *LET* in the i-th stage. Now, we are ready to describe the stopping conditions and the values F_i in both algorithms.

Algorithm $\texttt{LET}_\texttt{A}$, for A known before exploration
The value of F used in Algorithm *LET* for the first stage is $F_1 = \sqrt{2}/2$. The value of F for subsequent stages is given by $F_{i+1} = \frac{A}{k_i F_i}$. The stopping condition is $\{k^* F_i \geq 2A/F_i \text{ and } k^* F_i \geq P^* + 1\}$.

Algorithm $\texttt{LET}_\texttt{k}$, for k known before exploration
The value of F used in Algorithm *LET* for the first stage is $F_1 = \frac{1}{k+\sqrt{2}}$. The value of F for subsequent stages is given by $F_{i+1} = \min\left\{\frac{A_i}{kF_i}, \frac{\sqrt{2}}{2}\right\}$. The stopping condition is $\{A^*/F_i \geq 2kF_i \text{ and } A^*/F_i \geq P^* + 1 \text{ and } F_i < \sqrt{2}/2\}$.

Consider a moment t during the execution of Algorithm LET. Let $\mathcal{C}_t$ be the set of cells recorded as visited by Algorithm LET at moment t, and let $\mathcal{O}_t$ be the set of obstacles approached by the robot until time t. For each $C \in \mathcal{C}_t$, let B_C be the length of the intersection of the exterior boundary of cell C with the boundary of the terrain. For each $O \in \mathcal{O}_t$, let $|O|$ be the perimeter of obstacle O and let $k_t = |\mathcal{O}_t|$. The following proposition is proved similarly as Lemma 5.

Proposition 1. *There is a positive constant d such that the length of the trajectory of the robot until any time t, during the execution of Algorithm LET, is at most $d(\sum_{C\in\mathcal{C}_t}(B_C + A_C/F) + (k_t + 1) \cdot F + \sum_{O\in\mathcal{O}_t} |O|)$.*

The following lemma establishes the complexity of exploration if either the area of the terrain or the number of obstacles is known a priori.

Lemma 6. *Algorithm $\texttt{LET}_\texttt{A}$ (resp. $\texttt{LET}_\texttt{k}$) explores a terrain $\mathcal{T}$ of area A, perimeter P and with k obstacles, using a trajectory of length $O(P + A + \sqrt{Ak})$, if A (resp. k) is known before exploration.*

The following theorem shows that the lengths of trajectories in Lemma 6 and in Theorem 3 are asymptotically optimal.

Theorem 4. *Any algorithm for a robot with limited visibility, exploring polygonal terrains of area A, perimeter P and with k obstacles, produces trajectories of length $\Omega(P + A + \sqrt{Ak})$ in some terrains, even if the terrain is known to the robot.*

Lemma 6 and Theorem 4 imply

Theorem 5. *Consider terrains of area A, perimeter P and with k obstacles. If either A or k is known before the exploration, then the exploration of any such terrain can be performed using a trajectory of length $\Theta(P + A + \sqrt{Ak})$, which is asymptotically optimal.*

Notice that in order to explore a terrain at cost $O(P+A+\sqrt{Ak})$, it is enough to know the parameter A or k up to a multiplicative constant, rather than the exact value. This can be proved by a carefull modification of the proof of Lemma 6. For the sake of clarity, we stated and proved the weaker version of Lemma 6, with knowledge of the exact value.

Suppose now that no a priori knowledge of any parameters of the terrain is available. We iterate Algorithm $\texttt{LET}_\texttt{A}$ or $\texttt{LET}_\texttt{k}$ for A (resp. k) equal $1, 2, 4, 8, \ldots$ interrupting the iteration and doubling the parameter as soon as the explored area (resp. the number of obstacles seen) exceeds the current parameter value. The algorithm stops when the entire terrain is explored (which happens at the first probe exceeding the actual unknown value of A, resp. k). We get an exploration algorithm using a trajectory of length $O((P + A + \sqrt{Ak}) \log A)$, resp. $O((P+A+\sqrt{Ak}) \log k)$. By interleaving the two procedures we get the minimum of the two costs. Thus we have the following corollary.

Corollary 1. *Consider terrains of area A, perimeter P and with k obstacles. Exploration of any such terrain can be performed without any a priori knowledge at cost differing from the worst-case optimal cost with full knowledge only by a factor $O(\min\{\log A, \log k\})$.*

References

1. S. Albers and K. Kursawe and S. Schuierer, Exploring unknown environments with obstacles, Algorithmica 32 (2002), 123–143.
2. T. Bandyopadhyay, Z. Liu, M.H Ang, M.H, W.K.G Seah, Visibility-based exploration in unknown environment containing structured obstacles, Advanced Robotics (2005), 484-491.
3. E. Bar-Eli, P. Berman, A. Fiat and R. Yan, On-line navigation in a room, Journal of Algorithms 17 (1994), 319-341.
4. P. Berman, A. Blum, A. Fiat, H. Karloff, A. Rosen and M. Saks, Randomized robot navigation algorithms, Proc. 7th ACM-SIAM Symp. on Discrete Algorithms (1996), 74-84.
5. A. Blum, P. Raghavan and B. Schieber, Navigating in unfamiliar geometric terrain, SIAM Journal on Computing 26 (1997), 110-137.
6. N. Cuperlier, M. Quoy, C. Giovanangelli, Navigation and planning in an unknown environment using vision and a cognitive map, Proc. Workshop: Reasoning with Uncertainty in Robotics (2005), 48-53.
7. X. Deng, T. Kameda and C. H. Papadimitriou, How to learn an unknown environment, Proc. 32nd Symp. on Foundations of Comp. Sci. (FOCS 1991), 298–303.
8. X. Deng, T. Kameda and C. H. Papadimitriou, How to learn an unknown environment I: the rectilinear case, Journal of the ACM 45 (1998), 215-245.
9. Y. Gabriely, E. Rimon, Spanning-tree based coverage of continuous areas by a mobile robot, Proc. Int. Conf. of Robotics and Automaton (ICRA 2001), 1927-1933.
10. Y. Gabriely and E. Rimon, Competitive on-line coverage of grid environments by a mobile robot, Computational Geometry: Theory and Applications (2003), 24(3):197-224.
11. S.K. Ghosh, J.W. Burdick, A. Bhattacharya and S. Sarkar, Online algorithms with discrete visibility - exploring unknown polygonal environments, Robotics & Automation Magazine 15 (2008), 67-76.
12. F. Hoffmann. C. Icking, R. Klein and K. Kriegel, The polygon exploration problem, SIAM J. Comput. 31 (2001), 577–600.
13. C. Icking, T. Kamphans, R. Klein and E. Langetepe. Exploring an unknown cellular environment. In Abstracts of the 16th European Workshop on Computational Geometry, pages 140-143, 2000.
14. A. Kolenderska, A. Kosowski, M. Małafiejski and P. Żyliński. An Improved Strategy for Exploring a Grid Polygon, SIROCCO 2009, 222–236.
15. B. Kalyanasundaram and K. Pruhs, A Competitive Analysis of Algorithms for Searching Unknown Scenes, Comput. Geom. 3 (1993), 139-155.
16. S. Ntafos, Watchman routes under limited visibility, Comput. Geom. Theory Appl. 1 (1992), 149–170.
17. R. Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84:1182–1238, 1978.
18. C. H. Papadimitriou, M. Yannakakis, Shortest paths without a map, Theor. Comput. Sci. 84 (1991), 127–150.
19. R. Sim, J.J. Little, Autonomous vision-based exploration and mapping using hybrid maps and Rao-Blackwellised particle filters, Intelligent Robots and Systems (2006), 2082-2089.
20. B. Tovar, R. Murrieta-Cid, S. M. Lavalle, Distance-optimal navigation in an unknown environment without sensing distances, IEEE Transactions on Robotics 23 (2007), 506-518.